

Granger Independent Martingale Processes

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Abstract

We introduce a new class of processes for the evaluation of multivariate equity derivatives. The proposed setting is well suited for the application of the standard copula function theory to processes, rather than variables, and easily enables to enforce the martingale pricing requirement. The martingale condition is imposed in a general multidimensional Markov setting to which we only add the restriction of no-Granger-causality of the increments (Granger-independent increments). We call this class of processes GIMP (Granger Independent Martingale Processes). The approach can also be extended to the application of time change, under which the martingale restriction continues to hold. Moreover, we show that the class of GIMP processes is closed under time changing: if a Granger independent process is used as a multivariate stochastic clock for the change of time of a GIMP process, the new process is also GIMP.

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1 Introduction

Multivariate equity options are largely used both in structured finance and index-linked life insurance policies. Their payoffs depend on the application of an aggregation function to a set of underlying asset prices. Examples are altiplano notes that promise a payoff if all assets are above a given threshold, Everest notes that use the $\min(X_1, X_2, \dots, X_m)$ as the aggregation function, or basket options that use arithmetic average as aggregation function. Copula functions have been widely applied to the evaluation of these products. In the typical application, both in the industry and the literature, marginal distributions are calibrated on univariate option prices, and the copula is estimated from the time series of the underlying assets. The first proposals of these models go back to Rosenberg (1999) and Cherubini and Luciano (2002). Rosenberg(2003)

provided a risk neutral pricing model in a static setting. Finally, Van den Goorbergh et al. (2005) extended the model allowing for time varying dependence. Their approach is very close to ours, since they extend the no-arbitrage assumption to a dynamic setting by imposing a condition on the information structure that we call no-Granger causality in this paper.

The main contribution of this paper is to provide a systematic analysis of the properties of the class of processes used in Van den Goorbergh et al. (2005). We call these processes Granger-Independent Martingale Processes (GIMP). We will show that the time change technique can be applied to this class of processes, and that Granger independent processes are endowed with the same closure property as Lévy processes. Namely, if one uses a Granger-independent process as a stochastic clock for the time change of a Granger-independent process, he produces a process that is part of the same class. Moreover, the time change technique enables to extend the applicability of the model to a wider set of processes. One can in fact generate a Granger independent martingale process and apply an arbitrary stochastic clock to it, so that the martingale property is preserved even though the stochastic clock is not part of the class of Granger independent processes.

The paper is organized as follows. In section 2 we present a general multivariate arbitrage free model in the spirit of coupling marginal arbitrage free price processes. In section 3 we will address the issue of time change, proving that the class of GIMP is invariant under change of time. Section 4 concludes.

2 A multivariate model for price dynamics

We now describe the model for the market price dynamics of asset returns. Let $\{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{N}}, \mathbb{P}\}$ be the underlying filtered probability space. The setting is multivariate, so that we denote with X_t^j , $j = 1, 2, \dots, m$, the log-prices of assets in the economy at time t and with $\mathbf{X} = (X^1, \dots, X^m)$ the multidimensional stochastic process. Correspondingly, we denote with $(\mathcal{F}_t^j)_{t \in \mathbb{N}}$ the filtration collecting the information generated by the history of asset j up to time t and with $(\mathcal{F}_t^{\mathbf{X}})_{t \in \mathbb{N}}$ the filtration generated by the multidimensional stochastic process \mathbf{X} .

2.1 Granger independent processes

In the arbitrage-free pricing theory, it is required that the underlying discounted prices are martingale with respect to the filtration containing information generated by all the assets in the basket. In models based on the specification of the joint distribution (such as the multivariate generalization of the Black and Scholes model in the seminal Johnson, 1987 and Margrabe, 1978 papers) the martingale property is included from the very start, at the cost of making calibration of the marginal distributions more difficult. In copula-based models,

the martingale condition should be impounded in the model once the martingale marginal processes have been specified. We are going to show that this is linked to a hypothesis that is very well known in econometrics, and is called no-Granger causality. No-Granger-causality means that no information can help to predict the future values of a variable over and above its past history.

Definition 2.1. $X^1, \dots, X^{i-1}, X^{i+1}, \dots, X^m$ do not Granger cause X^i if and only if

$$\mathbb{P}[X_{t+1}^i \leq x | \mathcal{F}_t^X] = \mathbb{P}[X_{t+1}^i \leq x | \mathcal{F}_t^i]$$

for any $t \in \mathbb{N}$ and x , or, equivalently, in terms of increments if and only if

$$\mathbb{P}(\Delta X_t^k \leq x | \mathcal{F}_t^X) = \mathbb{P}(\Delta X_t^k \leq x | \mathcal{F}_t^k), \quad (1)$$

where $\Delta X_t^k = X_{t+1}^k - X_t^k$, for any $t \in \mathbb{N}$ and x .

The absence of Granger causality induces that, if the martingale property or the Markov property are satisfied by each process with respect to its own filtration, then they also hold with respect to the filtration generated by the whole multidimensional process. As for the martingale property, this is obvious since the no-Granger causality says that the distributions of X_{t+1}^i conditioned on \mathcal{F}_t^i and \mathcal{F}_t^X are the same.

Intuitively, (1) implies that, while the marginal conditional distribution of increments only depends on the corresponding marginal filtration, the conditional dependence structure of the vector of increments could depend on the whole filtration (\mathcal{F}_t^X).

The following is a particular specification of (1).

Definition 2.2. We say that a multidimensional stochastic process $\mathbf{X} = (X^1, \dots, X^m)$ has Granger independent increments if for any $k = 1, \dots, m$

$$\mathbb{P}(\Delta X_t^k \leq x | \mathcal{F}_t^X) = \mathbb{P}(\Delta X_t^k \leq x),$$

where $\Delta X_t^k = X_{t+1}^k - X_t^k$, $t \in \mathbb{N}$.

This notion of independence is in the middle between *componentwise independence* of increments

$$\mathbb{P}(\Delta X_t^k \leq x | \mathcal{F}_t^k) = \mathbb{P}(\Delta X_t^k \leq x),$$

and *vector independence* of increments

$$\mathbb{P}(\Delta X_t^1 \leq x_1, \dots, \Delta X_t^m \leq x_m | \mathcal{F}_t^X) = \mathbb{P}(\Delta X_t^1 \leq x_1, \dots, \Delta X_t^m \leq x_m).$$

While vector independence requires that the *joint* distribution of increments be independent of the levels, Granger independence only requires that the *marginal* distribution of the increments be independent of them.

Thanks to Sklar's theorem (see Nelsen, 2006), it is now immediate to convince oneself that if the \mathbb{R}^m -valued stochastic process \mathbf{X} has Granger independent increments and the conditional copula of the vector of increments $(\Delta X_t^1, \dots, \Delta X_t^m)$ given \mathcal{F}_t^X is independent of \mathcal{F}_t^X , then the process exhibits vector independent increments. Moreover, it is likewise obvious that, if the conditional copula of the vector of increments $(\Delta X_t^1, \dots, \Delta X_t^m)$ given \mathcal{F}_t^X coincides with the copula of the vector of increments $(\Delta X_t^1, \dots, \Delta X_t^m)$ conditioned on $\mathbf{X}_t = (X_t^1, \dots, X_t^m)$, then the stochastic process \mathbf{X} is a multidimensional Markov process.

Multidimensional Granger independent processes are very well suited to construct multidimensional exponential martingales, that is to model assets prices with respect to the risk neutral pricing measure. In fact, each unidimensional exponential process, being endowed with independent log-increments, satisfies the martingale property with respect to its natural filtration, once it is normalized with respect to its mean. Now, thanks to no-Granger independence, the process automatically turns out to be a multidimensional martingale. This fact justifies the following definition

Definition 2.3. *We define Granger-independent martingale process (GIMP) a multivariate process with Granger independent log-increments in which each univariate process is a martingale with respect to its own natural filtration.*

The above approach has been already applied in literature in the case of multivariate Garch processes and an equivalent martingale probability introduced, see van der Goorberg et al. (2005). Here below we analyze the same case showing that the equivalent martingale change of probability preserves the no-Granger causality.

Example 2.1. Copulas for GARCH martingale processes. *In this example we analyze a specific model in which the increments of each process follow a GARCH dynamics.*

More precisely, let t from 0 to $N > 0$ and denote with Ω^j the set of all possible trajectories from time 0 to time N of the process X^j . Let $\Omega = \Omega^1 \times \dots \times \Omega^m$ be the set of all multidimensional paths. \mathcal{F}_t^j and \mathcal{F}^X denote, as above, the filtration generated by the process X^j and by the whole market \mathbf{X} , respectively. We consider the probability space $(\Omega, \mathcal{F}_N^X, \mathbb{P})$ where \mathbb{P} denotes the objective probability and we call \mathbb{P}^j the projection of \mathbb{P} on $(\Omega^j, \mathcal{F}_N^j)$. As for the dynamics of the processes, we assume that the increment Y_t^j of each X^j from time $t-1$ to time t follows, with respect to \mathbb{P}^j , a dynamics of type

$$\begin{aligned} Y_t^j &= \mu_t^j - \frac{(H_t^j)^2}{2} + H_t^j Z_t^j \\ (H_t^j)^2 &= \omega_0^j + \omega_1^j (H_{t-1}^j)^2 + \omega_2^j (Y_{t-1}^j)^2 \\ Z_t^j &\sim N(0, 1) \text{ i.i.d.} \end{aligned}$$

with μ_t^j \mathcal{F}_{t-1}^j -adapted and ω_i^j positive constants for $i = 0, 1, 2$. Moreover we assume that

$$\begin{aligned} \mathbb{P}(Y_t^1 \leq y^1, \dots, Y_t^m \leq y^m | Y_1^1 = z_1^1, \dots, Y_{t-1}^1 = z_{t-1}^1, \dots, Y_1^m = z_1^m, \dots, Y_{t-1}^m = z_{t-1}^m) = \\ = C_{t|Y_1^1=z_1^1, \dots, Y_{t-1}^1=z_{t-1}^1, \dots, Y_1^m=z_1^m, \dots, Y_{t-1}^m=z_{t-1}^m}(F_{t-1}^1(y^1), \dots, F_{t-1}^m(y^m)) \end{aligned}$$

where $F_{t-1}^j(y^j) = \mathbb{P}^j(Y_t^j \leq y^j | Y_1^j = z_1^j, \dots, Y_{t-1}^j = z_{t-1}^j)$ and where we suppose that each conditional copula

$$C_{t|Y_1^1=z_1^1, \dots, Y_{t-1}^1=z_{t-1}^1, \dots, Y_1^m=z_1^m, \dots, Y_{t-1}^m=z_{t-1}^m}(u_1, \dots, u_m)$$

has a strictly positive density in $(0, 1)^m$.

This model clearly satisfies the no-Granger assumption.

We now introduce the martingale restriction on the marginal processes, following the general approach in Christoffersen, Elkamhi and Feunou (CEF, 2010), for which we construct the multivariate extension. For the sake of simplicity, we assume zero risk-free rate. CEF (2010) show that for each marginal process j there exists an equivalent probability $\mathbb{Q}^j \sim \mathbb{P}^j$ with respect to which the increments of the log-prices satisfy

$$\begin{aligned} Y_t^j &= -\frac{(H_t^j)^2}{2} + H_t^j \tilde{Z}_t^j \\ (H_t^j)^2 &= \omega_0^j + \omega_1^j (H_{t-1}^j)^2 + \omega_2^j (Y_{t-1}^j)^2 \\ \tilde{Z}_t^j &\sim N(0, 1) \text{ i.i.d.} \end{aligned}$$

and, then $S_t^j = e^{X_t^j}$ is an \mathcal{F}_t^j -martingale.

We now define the joint probability \mathbb{Q} on $(\Omega, \mathcal{F}_N^X)$ as

$$\mathbb{Q}(Y_t^1 \leq y^1, \dots, Y_t^m \leq y^m | \mathcal{F}_{t-1}^X) = C_{t|\mathcal{F}_{t-1}^X}(G_{t-1}^1(y^1), \dots, G_{t-1}^m(y^m))$$

where $G_{t-1}^j(y^j) = \mathbb{Q}^j(Y_t^j \leq y^j | Y_1^j = z_1^j, \dots, Y_{t-1}^j = z_{t-1}^j)$.

\mathbb{Q} is equivalent to \mathbb{P} and the distribution with respect to \mathbb{Q} of the multidimensional process \mathbf{X} continues to satisfy the no-Granger causality assumption. Since each S_t^j is a \mathcal{F}_t^j -martingale with respect to \mathbb{Q} , then it is a \mathcal{F}_t^X -martingale as well.

3 Time Changed Multidimensional Martingale Processes

In this section we will consider the case of multidimensional time-changed stochastic processes. We will develop the time change approach in two directions. The first one aims at providing a technique to build martingales with respect to the filtration generated by the whole multidimensional process, given that they satisfy the martingale property with respect to their natural filtration. The

second direction gives a technique to construct new Granger independent increments processes: this is obtained assuming that the processes have stationary increments and that the stochastic clock has Granger independent increments as well.

In what follows, we will represent the multidimensional stochastic clock as an \mathbb{R}^m -valued process $\mathbf{T} = (\mathbf{T}_t)_{t \in \mathbb{N}} = (T_t^1, \dots, T_t^m)_{t \in \mathbb{N}}$ such that $\mathbf{T}_0 = (0, \dots, 0)$ and such that each component T^j is an increasing process.

Given an \mathbb{R}^m -valued stochastic process $\mathbf{Y} = (\mathbf{Y}_t)_{t \in \mathbb{N}} = (Y_t^1, \dots, Y_t^m)_{t \in \mathbb{N}}$ we denote with $\mathbf{Y}_T = (Y_{T^1}^1, \dots, Y_{T^m}^m)$ the corresponding time changed process whose components are the stochastic processes $(Y_{T_t^j}^j)_{t \in \mathbb{N}}$.

Lemma 3.1. *Let $\mathbf{X} = (X^1, \dots, X^m)$ be a multidimensional stochastic process such that each one-dimensional stochastic process is not Granger caused by the others and \mathbf{T} be a multidimensional stochastic clock. We assume \mathbf{T} independent of \mathbf{X} . Then*

$$\mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x | \mathcal{F}_s^{X^T, T}) = \mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x | \mathcal{F}_s^{X_{T^j}^j, T}).$$

where $(\mathcal{F}_s^{X^T, T})$ is the filtration generated by the $2m$ -dimensional stochastic process $(\mathbf{X}_{T_s}, \mathbf{T}_s)$ and $(\mathcal{F}_s^{X_{T^j}^j, T})$ that generated by the $m+1$ -dimensional process $(X_{T_s^j}^j, \mathbf{T}_s)$.

Proof. For $s \in \mathbb{N}$ we have

$$\begin{aligned} \mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x | \mathbf{X}_{\mathbf{T}_r} = \mathbf{x}_r, \mathbf{T}_r = \mathbf{t}_r, r \in \mathbb{N}, r \leq s) &= \\ &= \int_{t_s^j}^{+\infty} \mathbb{P}(X_v^j - X_{t_s^j}^j \leq x | X_{t_r^j}^j = x_r^j, r \in \mathbb{N}, r \leq s) \\ & d\mathbb{P}(T_{s+1}^j \leq v | \mathbf{T}_r = \mathbf{t}_r, r \in \mathbb{N}, r \leq s) \end{aligned}$$

On the other hand

$$\begin{aligned} \mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x | X_{T_r^j}^j = x_r^j, \mathbf{T}_r = \mathbf{t}_r, r \in \mathbb{N}, r \leq s) &= \\ &= \int_{t_s^j}^{+\infty} \mathbb{P}(X_v^j - X_{t_s^j}^j \leq x | X_{t_r^j}^j = x_r^j, r \in \mathbb{N}, r \leq s) d\mathbb{P}(T_{s+1}^j \leq v | \mathbf{T}_r = \mathbf{t}_r, r \in \mathbb{N}, r \leq s) \end{aligned}$$

and the thesis follows. \square

Remark 3.1. *If we assume that the multidimensional process \mathbf{X} has Granger independent increments it is trivial to show that the conclusion is*

$$\mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x | \mathcal{F}_s^{X^T, T}) = \mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x | \mathcal{F}_s^T). \quad (2)$$

Proposition 3.1. *Let \mathbf{S} be a multivariate process such that each component is a positive martingale with respect to its natural filtration and it is not Granger caused by the others. Let \mathbf{T} be a multidimensional stochastic clock independent of \mathbf{S} . Then, the time changed stochastic process $\mathbf{S}_{\mathbf{T}}$ is a martingale with respect to its natural filtration.*

Proof. Let $\mathbf{X}_t = (\ln(S_t^1), \dots, \ln(S_t^m))$.

By the hypotheses, for all $v \in \mathbb{N}$, $\mathbb{E} \left[e^{X_{v+1}^j - X_v^j} \middle| \mathcal{F}_v^j \right] = 1$ and thanks to the above Lemma, if $s \in \mathbb{N}$,

$$\mathbb{E} \left[S_{T_{s+1}^j}^j - S_{T_s^j}^j \middle| \mathcal{F}_s^{X_T, T} \right] = S_{T_s^j}^j \mathbb{E} \left[e^{X_{T_{s+1}^j}^j - X_{T_s^j}^j} - 1 \middle| \mathcal{F}_s^{X_T, T} \right] = S_{T_s^j}^j \mathbb{E} \left[e^{X_{T_{s+1}^j}^j - X_{T_s^j}^j} - 1 \middle| \mathcal{F}_s^{X_{T^j}, T} \right].$$

Since

$$\begin{aligned} & \mathbb{E} \left[e^{X_{T_{s+1}^j}^j - X_{T_s^j}^j} - 1 \middle| X_{T_r^j}^j = x_r^j, \mathbf{T}_r = \mathbf{t}_r, r \in \mathbb{N}, r \leq s \right] = \\ &= \int_{t_s^j}^{+\infty} \mathbb{E} \left[e^{X_u^j - X_{t_s^j}^j} - 1 \middle| X_{T_r^j}^j = x_r^j, r \in \mathbb{N}, r \leq s \right] d\mathbb{P}(T_{s+1}^j \leq u | \mathbf{T}_r = \mathbf{t}_r, r \in \mathbb{N}, r \leq s) = 0, \end{aligned}$$

each $S_{T_j}^j$ is a martingale with respect to the filtration $(\mathcal{F}_t^{X_T, T})$ and hence it is a martingale with respect to the smaller filtration $(\mathcal{F}_t^{X_T})$ as required. \square

Until now we have studied under which conditions the martingale property is preserved through the time change. A natural question now takes us to the second question. What kind of dependence will be satisfied by the increments of the time changed process? We see below that, given that X has Granger independent increments, adding the assumption of marginal stationary increments and of a Granger independent clock yields the result and that under these assumptions the time changed process exhibits Granger independent increments.

Proposition 3.2. *Let $\mathbf{X} = (X^1, \dots, X^m)$ be a multidimensional stochastic process with Granger independent and stationary marginal increments and \mathbf{T} be a multidimensional stochastic clock with Granger-independent increments independent of \mathbf{X} . Then the time changed process $\mathbf{X}_{\mathbf{T}}$ has Granger independent increments.*

Proof. We start showing that, for $s \in \mathbb{N}$,

$$\mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x | \mathcal{F}_s^T) = \mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x) \quad (3)$$

In fact,

$$\begin{aligned}
& \mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x | \mathbf{T}_r = \mathbf{t}_r, r \in \mathbb{N}, r \leq s) = \\
& = \int_0^{+\infty} \mathbb{P}(X_{t_s^j+u}^j - X_{t_s^j}^j \leq x | T_{s+1}^j - T_s^j = u, \mathbf{T}_r = \mathbf{t}_r, r \in \mathbb{N}, r \leq s) \cdot \\
& \cdot d\mathbb{P}(T_{s+1}^j - T_s^j \leq u | \mathbf{T}_r = \mathbf{t}_r, r \in \mathbb{N}, r \leq s) = \\
& = \int_0^{+\infty} \mathbb{P}(X_{t_s^j+u}^j - X_{t_s^j}^j \leq x) dF_{T_{s+1}^j - T_s^j}(u) = \int_0^{+\infty} \mathbb{P}(X_u^j \leq x) dF_{T_{s+1}^j - T_s^j}(u),
\end{aligned}$$

thanks to marginal stationarity and

$$\begin{aligned}
& \mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x) = \\
& = \int_0^{+\infty} \int_0^{+\infty} \mathbb{P}(X_{v+u}^j - X_v^j \leq x | T_{s+1}^j - T_s^j = u, T_s^j = v) dF_{T_{s+1}^j - T_s^j, T_s^j}(u, v) = \\
& = \int_0^{+\infty} \int_0^{+\infty} \mathbb{P}(X_{v+u}^j - X_v^j \leq x) dF_{T_{s+1}^j - T_s^j}(u) dF_{T_s^j}(v) = \int_0^{+\infty} \mathbb{P}(X_u^j \leq x) dF_{T_{s+1}^j - T_s^j}(u)
\end{aligned}$$

where $F_{T_{s+1}^j - T_s^j, T_s^j}(u, v)$ denotes the joint cumulative distribution function of $(T_{s+1}^j - T_s^j, T_s^j)$ and $F_{T_{s+1}^j - T_s^j}(u)$ the cumulative distribution function of $T_{s+1}^j - T_s^j$.

Hence (3) is proved.

Now, by (2) and by (3)

$$\mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x | \mathcal{F}_s^{X_T}) = \mathbb{E} \left[\mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x | \mathcal{F}_s^{X_T, T}) \middle| \mathcal{F}_s^{X_T} \right] = \mathbb{P}(X_{T_{s+1}^j}^j - X_{T_s^j}^j \leq x)$$

and this trivially implies that X_T has Granger independent increments. \square

4 Conclusion

In this paper we address the main flaw of copula function applications to the evaluation of multivariate equity derivatives, namely the fact that copula functions are intrinsically static objects that are used to link variables, rather than processes. As a result, in standard copula functions applications it is impossible to impose consistency relationships among prices of the same product for different maturities. The consistency required has to do with the martingale condition and the dependence structure among the levels. Particularly, the dependence structure of the levels at different time horizons must be made consistent with the dependence structure of the increments, which provides the characteristic feature of any multivariate stochastic process.

Here we propose and study a new class of processes for which copula functions can be applied ensuring the intertemporal consistency of prices. The requirements imposed to this class of processes are that no process can be Granger-caused by any of the others, and that each process is an univariate martingale

with independent increments: this class of processes is called GIMP (Granger Independent Martingale Processes). The impact of a stochastic clock on the Granger causality assumption is analyzed and conditions under which this is preserved introduced.

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